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On light-cone SFT contact terms in a plane wave

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Abstract: Testing the BMN correspondence at non-zero string coupling g_s requires a one-loop string field theory calculation. At order g_s^2, matrix elements of the light-cone string field theory hamiltonian between single-string states receive two contributions: the iterated cubic interaction, and a contact term $\{Q, \overline{Q}\}$ whose presence is dictated by supersymmetry. In this paper we calculate the leading large $\mu p^+ \alpha'$ contribution from both terms for the set of intermediate states with two string excitations. We find precise agreement with the basis-independent order g_2^2 results from gauge theory.

Keywords: String Field Theory, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Berenstein, Maldacena and Nastase have recently demonstrated [1] that there is a natural correspondence between the states of type-IIB string theory in a plane-wave background and a class of operators in $\mathcal{N} = 4$ SU(N) gauge theory with large R-charge J. This correspondence involves a limit of the gauge theory in which the quantities

$$\lambda' = \frac{g^2_{\rm YM} N}{J^2}, \qquad g_2 = \frac{J^2}{N} \tag{1.1}$$

and $g^2_{\rm YM}$ are held fixed while N and J are taken to infinity. Moreover, the relation

$$\frac{2}{\mu} P^- = \Delta - J \tag{1.2}$$

was verified in the free theory ($g_2 = 0$), originally to order λ' [1], and to all orders in λ' [2, 3]. Here P^- is the light-cone hamiltonian in string theory, and Δ is the generator of scale transformations in the gauge theory.

Since P^- is a symmetry generator in a background which is not corrected by interactions, it is natural to expect that the relation (1.2) continues to hold in the presence of string interactions, as proposed in [4]. Light-cone string field theory in the plane wave background has been constructed in [5, 6] and developed further in [7]–[14], generalizing the flat space construction of [15, 16]. On the gauge theory side, a number of papers have

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addressed the calculation of non-planar diagrams [4, 17]–[22]. However, it is important to note that despite some suggestive hints, there has so far been *absolutely no definitive evidence* that (1.2) holds for $g_2 \neq 0$. While this may seem like a strong statement, it is based on the fact that at order g_2 the matching of the matrix elements on the two sides of equation (1.2) fixes the state-operator map and thus cannot be used also as a test.

The single- and double-trace BMN operators in the gauge theory cannot be identified with single- and double-string states at finite g_2 because the former are not orthogonal [4, 21, 22]. Therefore we cannot check the relation (1.2) at the level of *matrix elements* without first identifying the g_2-dependent basis transformation between BMN operators and string states. Instead, the BMN correspondence only requires that $\frac{2}{\mu}P^-$ and $\Delta - J$ have the same *eigenvalues*. On the gauge theory side, the eigenvalues of the anomalous dimension matrix have been calculated to order g_2^2 within the subspace of BMN operators with two scalar impurities [21, 22], with the result

$$(\Delta - J)_n = 2 + \lambda' \left[n^2 + \frac{g_2^2}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32\pi^2 n^2} \right) \right]. \tag{1.3}$$

This result was recovered in [12] using a combination of string bit model predictions and $\mathcal{O}(g_2)$ input from string field theory computations, but an honest string field theory calculation of the $\mathcal{O}(g_2^2)$ term in (1.3) is so far lacking.

One can also turn the problem around and construct the state-operator map so as to ensure that the equation (1.2) is valid at the level of matrix elements [4]. It is important to stress that this approach has the same predictive power as the one above, since the existence of an operator basis in gauge theory such that (1.2) is valid at the level of matrix elements is equivalent to the equality of eigenvalues of the two operators. The essential test of the proposal (1.2) and, more generally, of the BMN conjecture comes from comparing the matrix elements of other operators once the state-operator map is fixed. A proposal for the state-operator map to all orders in g_2 has been recently put forward in [12].

In this paper we compute matrix elements of the string field theory hamiltonian P^- at order g_s^2 in the limit of large $\mu p^+ \alpha'$ between two single-string states for the same set of intermediate states used in gauge theory. Recall that λ' and g_2 are related to the string theory parameters by $\lambda' = (\mu p^+ \alpha')^{-2}$ and $g_2 \lambda' = 4\pi g_s$.

The $\mathcal{O}(g_2^2)$ matrix element between single-string states is a one-loop-like effect, which enters the hamiltonian through supersymmetry via $P^- = \frac{1}{2}\{Q, \overline{Q}\}$. This calculation is performed using techniques analogous to those used in string loop diagrams although we include only those intermediate states which have one bosonic excitation and one fermionic excitation. These are the states corresponding to the perturbative gauge theory calculations.

In the following section we review the procedure for determining the state-operator correspondence at finite g_2 and write the known result at order g_2. In section 3 we review the large $\mu p^+ \alpha'$ limit of light-cone string field theory in the plane wave background and calculate the two diagrams (see figure 1) which contribute to the eigenvalues (1.3). We conclude with a detailed discussion of the assumptions made in our work and some interesting questions raised by our analysis.

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2. BMN correspondence at finite g_2

String field theory has a natural basis of single- (double-, etc.) string states, and the gauge theory has a natural basis of single- (double-, etc.) trace operators. As mentioned in the introduction, the natural identification between these two bases breaks down at order g_2 because the scalar product is orthogonal in the string theory basis but not in the gauge theory basis [4, 21, 22]. The explicit transformation between these two bases can be worked out as follows [4]. First one diagonalizes the scalar product in the gauge theory, which fixes the basis transformation up to an arbitrary orthogonal transformation. This remaining ambiguity is fixed by requiring the matrix elements of (1.2) to agree. At the end of the day, this procedure is equivalent to diagonalizing both sides of (1.2) and then using the eigenvectors to work backwards and determine the basis transformation.

The state-operator map has been worked out to order g_2 in [4, 12, 23]. In order to write the transformation in a succinct form, let us denote by $|1,n\rangle$ the normalized state corresponding to the single-trace BMN operator $\sum_l e^{2\pi inl/J}\,\mathrm{Tr}(\phi Z^l \psi Z^{J-l})$, and by $|2,y\rangle$ and $|2,n,y\rangle$ respectively the normalized states corresponding to the double-trace BMN operators $\mathrm{Tr}(\phi Z^{J_1})\,\mathrm{Tr}(\psi Z^{J-J_1})$ and $\sum_l e^{2\pi inl/J_1}\,\mathrm{Tr}(\phi Z^l \psi Z^{J_1-l})\,\mathrm{Tr}(Z^{J-J_1})$, where $y = J_1/J$. Then we introduce the splitting-joining operator Σ (see [24, 25] for details), whose action on single-string states is

$$\Sigma|1,n\rangle = \sum_{m,y}\sqrt{\frac{1-y}{Jy}}\frac{\sin^2(\pi ny)}{\pi^2(n-m/y)^2}|2,m,y\rangle - \sum_y \frac{\sin^2(\pi ny)}{\sqrt{J}\pi^2 n^2}|2,y\rangle\,. \tag{2.1}$$

The transformation between the gauge theory basis $|\psi\rangle$ and the string theory basis $|\widetilde{\psi}\rangle$ may then be written as

$$|\widetilde{\psi}\rangle = |\psi\rangle - \frac{g_2}{2}\Sigma|\psi\rangle + \mathcal{O}(g_2^2)\,. \tag{2.2}$$

As expected, a single-string state corresponds to a single-trace operator with an order g_2 admixture of double-trace operators. In [12], it was conjectured that the basis transformation takes the form

$$|\widetilde{\psi}\rangle = e^{-g_2\Sigma/2}|\psi\rangle\,, \tag{2.3}$$

to all orders in g_2.

The basis transformation (2.2) was chosen in part to ensure that the matrix elements of (1.2) agree at order g_2, so we need to go to next order to get a nontrivial check of the BMN correspondence. The order g_2^2 matrix elements of $\Delta - J$ in the conjectured string basis $|\widetilde{\psi}\rangle$ given by equation (2.3) are [12]

$$\langle\widetilde{1},m,\pm|(\Delta-J)_{(2)}|\widetilde{1},n,\pm\rangle = \frac{\lambda' g_2^2}{16\pi^2}(B_{mn} \pm B_{m,-n})\,. \tag{2.4}$$

Here the subscript (2) denotes the order g_2^2 part, the matrix B_{mn} is defined in appendix B (it corresponds to non-nearest neighbor interaction diagrams in gauge theory), and we have introduced

$$|\widetilde{1},n,\pm\rangle = \frac{1}{\sqrt{2}}\left(|\widetilde{1},n\rangle \pm |\widetilde{1},-n\rangle\right)\,. \tag{2.5}$$

The matrix element (2.4) vanishes between $|+\rangle$ and $|-\rangle$ states because $|+\rangle$ is in the **9** representation of SO(4) while $|-\rangle$ is in the **6**.

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Still (2.4) does not obviously give a nontrivial check of the BMN correspondence, because the matrix element is sensitive to the order g_2^2 part of the basis transformation (2.2), and in writing (2.4) we used the conjecture (2.3) to order g_2^2. However, it was shown in [4] that the diagonal matrix elements ($m = n$) are insensitive to the order g_2^2 freedom in adjusting the basis transformation. This is because an order g_2^2 transformation in (2.2) shifts the matrix elements of (2.4) by an amount proportional to the difference between the order g_2^0 energies of the in- and out-states, which vanishes if $m = \pm n$. However, the off-diagonal matrix elements in (2.4) can be set to any desired value by choosing the order g_2^2 term in (2.2) appropriately. The possibility of performing such transformations is essentially due to the fact that the gauge theory analog of splitting and joining of strings is not corrected at order g_2^2.

In summary: if the string field theory calculation fails to reproduce the off-diagonal terms in (2.4), this is merely an indication that the basis transformation (2.3) needs to be adjusted at order g_2^2. On the other hand, if the string field theory calculation fails to reproduce the diagonal terms in (2.4), then something must be seriously wrong.

3. Light-cone superstring field theory

In light-cone string field theory we study the dynamical symmetry generators P^- and Q expanded in powers of the string coupling g_s,

$$P^- = P^-_{(0)} + g_s P^-_{(1)} + g_s^2 P^-_{(2)} + \cdots , \qquad Q = Q_{(0)} + g_s Q_{(1)} + \cdots . \tag{3.1}$$

The leading interactions $P^-_{(1)}$ and $Q_{(1)}$, which are cubic in string fields and mediate simple string joining and splitting, were determined in [5] for the plane wave background.

In this paper we study $P^-_{(2)}$ with the aim of reproducing the matrix elements (2.4) calculated in the gauge theory. All of the terms in (3.1) are determined in principle by requiring closure of the plane wave supersymmetry algebra to all orders in g_s. At second order, we have the relation

$$2P^-_{(2)} = \{Q_{(1)}, \overline{Q}_{(1)}\} + \{Q_{(0)}, \overline{Q}_{(2)}\} + \{Q_{(2)}, \overline{Q}_{(0)}\}. \tag{3.2}$$

Although $Q_{(2)}$ is not known, it does not contribute to the order g_s^2 matrix elements of (3.2) between single-string states since $Q_{(2)}$ is quartic in string fields at tree level. Therefore we are only interested in the first term in (3.2). Note that the matrix element between single-string states has not been computed to order g_s^2 even in flat space, although for a discussion of two-string state matrix elements at this order see [16, 26, 27, 28]-[29]

In supersymmetric light-cone string field theory the eigenvalues (1.3) receive the two contributions

$$\frac{1}{2}\int_0^\infty dT\, \langle \tilde{1}, n| P^-_{(1)} e^{T(P^-_{(0)} - E_n)} P^-_{(1)} |\tilde{1}, n\rangle + \frac{1}{2}\langle \tilde{1}, n|\{Q_{(1)}, \overline{Q}_{(1)}\}|\tilde{1}, n\rangle\,, \tag{3.3}$$

corresponding to the two diagrams in figure 1. The numerical factor in the first term is due to the reflection symmetry of the first diagram in that figure, while the numerical factor in the second term arises from equation (3.2).

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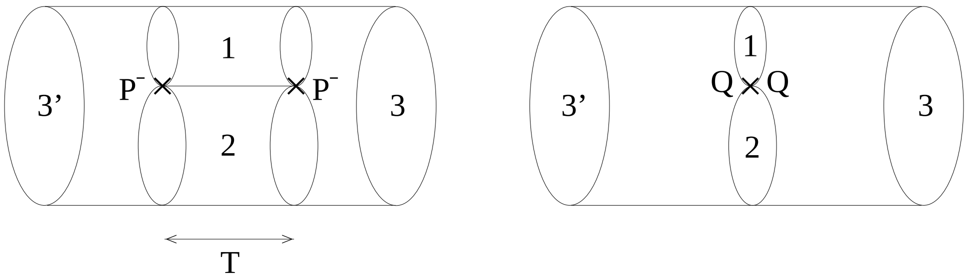

Figure 1: The two light-cone diagrams which contribute at order g_s^2 to the eigenvalue (1.3). The first is the iterated cubic interaction $P^-_{(1)}$, while the second is the quadratic contact term induced by $\{Q_{(1)}, Q_{(1)}\}$.

In this paper we will calculate the two terms in (3.3) by first taking the large $\mu p^+ \alpha'$ limit of $P^-_{(1)}$ and $Q_{(1)}$, and then including in the sum over intermediate states only a particular class of states — namely, those string states which have only two bosonic excitations (for $P^-_{(1)}$) or one bosonic and one fermionic excitation (for $Q_{(1)}$). The order of limits problem is one of the poorly understood aspects of the BMN correspondence [7]. This order of limits coincides with the order of limits which has been used in the gauge theory calculation of (1.3). Namely, $\Delta - J$ has been diagonalized order by order in λ within the subspace of two-impurity BMN operators, not order by order in λ' within the space of all operators. Therefore, in order to compare with (1.3), our string theory calculation will include only intermediate states with two impurities.

3.1 Dynamical generators

The matrix elements of the dynamical generators in the plane wave background were presented in [5] to first order in g_s up to an overall function $v(\mu, \alpha_3, y)$, where $\alpha_{(r)} = 2p^+_{(r)}$ and $y = -\alpha_{(1)}/\alpha_{(3)}$. They can be expressed as states in the three-string Fock space as

$$\begin{aligned} |P^-\rangle &= \left[(K_+^I + K_-^I)(K_+^J - K_-^J) - \frac{1}{2}\mu\alpha\delta^{IJ} \right] v_{IJ}|V\rangle \,, \\ |Q_{\dot a}\rangle &= (K_+^I s_{I\dot a} + iK_-^I t_{I\dot a})|V\rangle \,, \\ |\overline{Q}_{\dot a}\rangle &= (K_+^I t_{I\dot a} - iK_-^I s_{I\dot a})|V\rangle \,. \end{aligned} \tag{3.4}$$

We refer the reader to the papers [5, 6, 10] for details, since we will present only the parts of (3.4) which are relevant to our calculation.

The vertex $|V\rangle$ is given by

$$|V\rangle = \exp\left[\frac{1}{2} \sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} a^{I\dagger}_{m(r)} \overline{N}^{(rs)}_{mn} a^{J\dagger}_{n(s)} \delta_{IJ} \right] E_b E_b^0 |0\rangle|0\rangle|0\rangle \,, \tag{3.5}$$

where E_b is exponential in fermionic creation operators

$$E_b = \exp\left[\sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b^\dagger_{-m(r)} Q^{(rs)}_{mn} b^\dagger_{n(s)} - \sqrt{2}\Lambda \sum_{r=1}^{3} \sum_{m=1}^{\infty} Q^{(r)}_m b^\dagger_{-m(r)} \right] \tag{3.6}$$

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and E_b^0 is linear in the fermionic zero modes,

$$E_b^0 = \prod_{a=1}^{8}(\lambda^a_{(1)} + \lambda^a_{(2)} + \lambda^a_{(3)})\,. \tag{3.7}$$

The tensors s, t and v are defined by

$$\begin{aligned}
v^{IJ} &= \delta^{IJ} - \frac{i}{\alpha}\gamma^{IJ}_{ab}Y^aY^b + \frac{1}{6\alpha^2}\gamma^{IK}_{ab}\gamma^{JK}_{cd}Y^aY^bY^cY^d - \\
&\quad - \frac{4i}{6!\alpha^3}\gamma^{IJ}_{ab}\epsilon_{abcdefgh}Y^cY^dY^eY^fY^gY^h + \frac{16}{8!\alpha^4}\delta^{IJ}\epsilon_{abcdefgh}Y^aY^bY^cY^dY^eY^fY^gY^h\,,\\
s_{I\dot{a}} &= 2\gamma^I_{\dot{a}a}Y^a - \frac{8}{6!\alpha^2}\gamma^{IJ}_{ab}\gamma^J_{c\dot{a}}\epsilon^{abcdefgh}Y^dY^eY^fY^gY^h\,,\\
t_{I\dot{a}} &= -\frac{2}{3\alpha}\gamma^{IJ}_{ab}\gamma^J_{c\dot{a}}Y^aY^bY^c - \frac{16}{7!\alpha^3}\gamma^I_{a\dot{a}}\epsilon^{abcdefgh}Y^bY^cY^dY^eY^fY^gY^h\,,
\end{aligned} \tag{3.8}$$

where $\alpha = \alpha_{(1)}\alpha_{(2)}\alpha_{(3)}$. Finally, the quantities $K_\pm$ and Y are linear in creation operators,

$$\begin{aligned}
K^I_+ &= \sum_{r=1}^{3}\sum_{m=0}^{\infty}F^+_{m(r)}a^{I\dagger}_{m(r)}\,, \qquad K^I_- = \sum_{r=1}^{3}\sum_{m=1}^{\infty}F^-_{m(r)}a^{I\dagger}_{-m(r)}\,,\\
Y &= Y_0(\alpha_{(1)}\lambda_{(2)} - \alpha_{(2)}\lambda_{(1)}) + \sum_{r=1}^{3}\sum_{m=1}^{\infty}Y_{m(r)}b^\dagger_{m(r)}\,.
\end{aligned} \tag{3.9}$$

3.2 Dynamical generators at large $\mu p^+\alpha'$

The matrix elements $\overline{N}^{(rs)}_{mn}$, $F^\pm_{m(r)}$, $Y_{m(r)}$, $Q^{(rs)}_{mn}$ and $Q^{(r)}_m$ are known for all $\mu p^+\alpha'$. In this subsection we write the leading behavior of these quantities for large $\mu p^+_{(3)}\alpha'$. In this limit the nonzero Neumann matrices are[1]

$$\overline{N}^{(13)}_{mn} = (-1)^{m+n}\frac{2}{\pi}\frac{y^{\frac{3}{2}}n\sin\pi ny}{m^2-n^2y^2}\,, \qquad \overline{N}^{(23)}_{mn} = (-1)^{n+1}\frac{2}{\pi}\frac{(1-y)^{\frac{3}{2}}n\sin\pi ny}{m^2-n^2(1-y)^2}\,, \tag{3.10}$$

$$\overline{N}^{(13)}_{0n} = (-1)^{n+1}\frac{1}{\pi}\sqrt{\frac{2}{y}}\frac{\sin\pi ny}{n}\,, \qquad \overline{N}^{(23)}_{0n} = (-1)^{n}\frac{1}{\pi}\sqrt{\frac{2}{1-y}}\frac{\sin\pi ny}{n}\,, \tag{3.11}$$

and

$$\overline{N}^{(r3)}_{-m-n} = \frac{m}{n}\frac{\alpha_{(3)}}{\alpha_{(r)}}\overline{N}^{(r3)}_{mn}\,, \tag{3.12}$$

where $m,n > 0$ and $y = -\alpha_{(1)}/\alpha_{(3)}$. The fermionic Neumann matrices can be obtained similarly.

The leading matrix elements of the prefactor which will appear in the calculations of the next section are

$$\begin{aligned}
F^-_{n(3)} &= i(-1)^{n+1}\sqrt{\frac{2\alpha_{(1)}\alpha_{(2)}}{\pi}}\sin\pi ny\,,\\
Y_{m(1)} &= \frac{1+\Pi}{2}\sqrt{\frac{\alpha_{(2)}|\alpha_{(3)}|}{4\mu\pi}}(-1)^{m+1}\,,
\end{aligned}$$

[1] We set α'=2 here.

$$Y_{m(2)} = \frac{1+\Pi}{2}\sqrt{\frac{\alpha_{(1)}|\alpha_{(3)}|}{4\mu\pi}}\,,$$
$$Y_0 = \sqrt{\frac{|\alpha_{(3)}|}{4\mu\pi\alpha_{(1)}\alpha_{(2)}}}\left[4\pi\mu\frac{\alpha_{(1)}\alpha_{(2)}}{|\alpha_{(3)}|}\frac{1-\Pi}{2}+\frac{1+\Pi}{2}\right]. \tag{3.13}$$

For other bosonic components see [6, 14], and for fermionic [10].

3.3 Matrix element of $(P^-_{(1)})^2$

The first contribution to the eigenvalue (1.3) comes from the iterated $P^-_{(1)}$ interaction (the first term in (3.3)). This calculation was performed in [12] using the expressions for $P^-_{(1)}$ at large $\mu p^+\alpha'$ given in the previous subsection, with the result

$$\frac{2}{\mu}\left[\frac{1}{2}\langle\tilde{1},n,+|P^-_{(1)}\frac{1}{\Delta}P^-_{(1)}|\tilde{1},n,+\rangle\right] = \frac{1}{\mu}\sum_i \frac{\langle\tilde{1},n,+|P^-_{(1)}|i\rangle\langle i|P^-_{(1)}|\tilde{1},n,+\rangle}{E_n-E_i} = -\frac{g_2^2\lambda'}{8\pi^2}B_{n,-n}\,, \tag{3.14}$$

where, as discussed above, the sum runs over intermediate 2-string states with only two bosonic excitations (which can either be on the same string or different strings). Note that the matrix element on the right-hand side of (3.14) is expressed in terms of unit-normalized states (which are natural from the gauge theory perspective) rather than delta-function normalized states (see [19]). In writing the final result we chose to normalize $P^-_{(1)}$ to absorb a number of irrelevant factors by taking $v(\mu,\alpha_{(3)},y) = g_2/(\langle v^{II}\rangle 2\sqrt{2}\alpha^3_{(3)})$. Finally, note that the analogous calculation (3.14) for the state $|\tilde{1},n,-\rangle$ gives zero, since the matrix element of $P^-_{(1)}$ between $|\tilde{1},n,-\rangle$ and any two-impurity two-string state vanishes.

3.4 Matrix element of $\{\overline{Q}_{(1)},Q_{(1)}\}$

In this subsection we will calculate the matrix element $\{\overline{Q}_{\dot{a}},Q_{\dot{b}}\}$ (suppressing the subscript (1)) for an analogous class of intermediate states: those 2-string states which have only one bosonic excitation and one fermionic excitation. There are two possibilities. If the worldsheet mode number m is nonzero, then the impurities have to sit on the same string, so in the conventions of [5] we have

$$|1\rangle = \frac{1}{2}(a^{K\dagger}_m + ie(m)a^{K\dagger}_{-m})(b^{a\dagger}_m - ie(m)b^{a\dagger}_{-m})|\mathrm{v}\rangle\,, \qquad |2\rangle = |\mathrm{v}\rangle\,, \tag{3.15}$$

where $e(m) = \mathrm{sign}(m)$, $|\mathrm{v}\rangle$ is the ground-state of the world-sheet hamiltonian, and K and a are spinor indices that will be summed over all possible values for the intermediate states. When $m = 0$, the two oscillators $a^{K\dagger}_0$ and $b^{a\dagger}_0$ can sit on either the same string or different strings, and all possibilities are included in the sum over intermediate states.

The single-string states introduced in (2.5) may be written with the string oscillator conventions of [5] as

$$|\tilde{1},n,+\rangle = \frac{1}{\sqrt{2}}(a^{I\dagger}_n a^{J\dagger}_n + a^{I\dagger}_{-n}a^{J\dagger}_{-n})|\mathrm{v}\rangle\,,$$
$$|\tilde{1},n,-\rangle = \frac{1}{\sqrt{2}}(a^{I\dagger}_n a^{J\dagger}_{-n} - a^{I\dagger}_{-n}a^{J\dagger}_n)|\mathrm{v}\rangle\,. \tag{3.16}$$

It can be checked that only the terms with K_- in Q and $\overline{Q}$ are leading at large $\mu p^+\alpha'$. Moreover only the term Y^5 contributes in s_I and term Y^3 contributes in t_I.

With these clarifications, it is not hard to construct the matrix elements of the anti-commutator $\{\overline{Q}_{\dot a}, Q_{\dot b}\}$,

$$\langle \tilde{1}, n, + | \{\overline{Q}_{\dot a}, Q_{\dot b}\} | \tilde{1}, p, + \rangle = \frac{g_2^2 \delta_{\dot a \dot b}}{8\alpha_{(3)}^6} \int_0^1 \frac{dy}{y(1-y)} (F^-_{n(3)})^2 \sum_{m=1}^{\infty} \sum_{s=1,2} \overline{N}^{(s3)}_{-m-n} \overline{N}^{(s3)}_{-m-p} (Y_{m(s)})^2 \tag{3.17}$$

$$\begin{aligned}\langle \tilde{1}, n, - | \{\overline{Q}_{\dot a}, Q_{\dot b}\} | \tilde{1}, p, - \rangle &= \frac{g_2^2 \delta_{\dot a \dot b}}{8\alpha_{(3)}^6} \int_0^1 \frac{dy}{y(1-y)} (F^-_{n(3)})^2 \times \\ &\times \Bigg[\sum_{s=1,2} \sum_{m=1}^{\infty} \overline{N}^{(s3)}_{mn} \overline{N}^{(s3)}_{mp} (Y_{m(s)})^2 + \\ &+ 2 \sum_{s,s'=1,2} \frac{\alpha_{(3)} y^2 (1-y)^2}{\alpha_{(s')}} \overline{N}^{(s3)}_{0n} \overline{N}^{(s3)}_{0p} (Y_{0(s')})^2 \Bigg].\end{aligned} \tag{3.18}$$

It is not hard to see the origin of the various factors appearing in these equations. In (3.17), the bosonic excitation on the intermediate string must have negative mode number (in our basis, negative and non-negative modes do not couple). In (3.18), the second term involves a double sum over both strings because when the intermediate excitations are zero-modes they do not have to sit on the same string. Finally, the measure arises because in computing the matrix element above we inserted a complete set of (physical) 2-string states (with two excitations) between Q and $\overline{Q}$. We used the string theoretic normalization of states, $\langle p_i^+ | p_j^+ \rangle = p_i^+ \delta(p_i^+ - p_j^+)$. Thus, the term $|i\rangle|j\rangle\langle i|\langle j|$ in the identity operator for the 2-string states appears multiplied by $1/(p_i^+ p_j^+)$, together with an integral for each of the two momenta. Because of the p^+ conservation constraint for each matrix element one of the two p^+ integrations can be trivially performed and one is left with only one integral, with the measure displayed above. Finally, the overall coefficient arises due to certain relations between v^{IJ} and the derivative of $s^I_{\dot a}$ with respect to Y [5].

Using the sums from the appendix A as well as the choice of normalization of $P^-_{(1)}$, the expressions for Y_m and F_m from the previous section, we find for the diagonal pieces

$$\begin{aligned}\frac{2}{\mu} \langle \tilde{1}, n, + | \{\overline{Q}_{\dot a}, Q_{\dot b}\} | \tilde{1}, n, + \rangle &= \frac{g_2^2 \lambda'}{8\pi^2} \delta_{\dot a \dot b} \left(\frac{1}{3} + \frac{5}{8\pi^2 n^2} \right) = \frac{g_2^2 \lambda'}{8\pi^2} \delta_{\dot a \dot b} (B_{nn} + B_{n,-n}) \,, \\ \frac{2}{\mu} \langle \tilde{1}, n, - | \{\overline{Q}_{\dot a}, Q_{\dot b}\} | \tilde{1}, n, - \rangle &= \frac{g_2^2 \lambda'}{8\pi^2} \delta_{\dot a \dot b} \left(\frac{1}{3} + \frac{35}{8\pi^2 n^2} \right) = \frac{g_2^2 \lambda'}{8\pi^2} \delta_{\dot a \dot b} (B_{nn} - B_{n,-n}) \,.\end{aligned} \tag{3.19}$$

Extracting the matrix elements of P^- and adding them to (3.14), we find that the total matrix element matches the gauge theory result of [4, 21, 22]. Note that we have not attempted to match the overall coefficient, since the precise normalization of (3.4) in string field theory has not yet been fixed. The relative normalization between (3.14) and (3.19) is the same due to supersymmetry [12]. The fact that the n dependence of (3.19) works out is highly nontrivial. It is also clear that the normalization of Q cannot be independently adjusted, since it is related to the normalization of H by the order g_s supersymmetry algebra.

For the off-diagonal transition the result of the summation and integrations over y is

$$\frac{2}{\mu}\langle\tilde{1},n,+|\{\overline{Q}_{\dot{a}},\,Q_{\dot{b}}\}|\tilde{1},p,+\rangle = \frac{g_2^2\lambda'}{8\pi^2}\delta_{\dot{a}\dot{b}}\left[\frac{n^2+p^2}{\pi^2(n^2-p^2)^2}\right] = \frac{g_2^2\lambda'}{8\pi^2}\delta_{\dot{a}\dot{b}}(B_{np}+B_{n,-p})\,, \tag{3.20}$$
$$\frac{2}{\mu}\langle\tilde{1},n,-|\{\overline{Q}_{\dot{a}},\,Q_{\dot{b}}\}|\tilde{1},p,-\rangle = \frac{g_2^2\lambda'}{8\pi^2}\delta_{\dot{a}\dot{b}}\left[\frac{3n^4-4n^2p^2+3p^4}{\pi^2np(n^2-p^2)^2}\right] = \frac{g_2^2\lambda'}{8\pi^2}\delta_{\dot{a}\dot{b}}(B_{np}-B_{n,-p})\,.$$

This expression agrees with the off-diagonal elements in (2.4), giving the further support to the conjecture of [12].

4. Discussion and conclusion

In this note we have calculated order g_2^2 matrix elements of the string field theory hamiltonian between single-string states. The diagonal matrix elements are precisely those needed to recover the eigenvalues (1.3) of $\Delta - J$ which have been computed on the gauge theory side within the subspace of two-impurity BMN operators. Moreover, we find agreement between the off-diagonal matrix elements (3.20) and those predicted by the conjectured state-operator map (2.3), thereby confirming the proposal of [12] to order g_2^2.

It is important to stress that the eigenvalues (1.3) result from a truncated calculation on the gauge theory side. Specifically, the operator $\Delta - J$ has been diagonalized perturbatively, order by order in λ, within the subspace of two-impurity BMN operators. It has been stressed in [7, 12] that there is no reason why the large J limit of the λ perturbation series has to agree order by order with the λ' series. (Recall that the BMN limit requires taking the $\lambda \to \infty$ limit of quantities which can be calculated in the gauge theory only at small λ.) Therefore, in order to compare with the result (1.3), we have done a similar truncation of the string field theory calculation, by including only two-impurity intermediate states.

Matrix elements in which the number of impurities are not conserved have not yet been analyzed in the gauge theory, but on the string theory side it was pointed out in [6] that matrix elements in which two impurities are created or destroyed are actually larger, by a factor of $\mu p^+\alpha'$, than impurity-conserving matrix elements. Since $\mu p^+\alpha' = 1/\sqrt{\lambda'}$, it seems that these matrix elements cannot be seen in perturbation theory around $\lambda' = 0$. This would indeed be the case if λ' were the coupling constant order by order in perturbation theory. While this seems to be true for operators with $\Delta - J = 1$ impurities, it may happen that at some higher order these operators mix with ones with $\Delta - J \geq 2$ impurities. Then, perturbation theory will become an expansion in λ rather than λ'. Since the 't Hooft coupling is taken to be large, reliable results require resummation of the perturbation theory. Then, the appearance of $1/\sqrt{\lambda'}$ becomes a strong coupling effect, similar to the appearance of $\sqrt{\lambda}$ in the Wilson loop calculations in the context of the AdS/CFT correspondence. Unlike that case the eventual emergence of $1/\sqrt{\lambda'}$ should be signaled in perturbation theory by a divergence in the limit $J \to \infty$ at some loop order.

Both terms in (3.3) receive contributions from intermediate states with more than two creation operators. In the large $\mu p^+\alpha'$ limit, the only non-vanishing matrix elements in which the number of impurities are not conserved require changing the number of impurities by two. Each matrix element of this type is larger by a factor of $\mu p^+\alpha'$ compared to matrix elements in which the number of impurities is conserved. However, the energy denominator is also larger by a factor of $(\mu p^+\alpha')^2$ and thus these intermediate states contribute to leading order. It turns out that the contribution of these states is actually linearly divergent. This is related to the fact that the computation is done in the large $\mu p^+\alpha'$ limit. At finite $\mu p^+\alpha'$ this divergence is regularized. Since the initial divergence was linear, taking the large $\mu p^+\alpha'$ limit at the end leads to a contribution of order $\mu p^+\alpha'$ to the masses of string states. The gauge theory counterpart of this effect is a $1/\sqrt{\lambda'}$ contribution to the anomalous dimension of some (appropriately redefined) operators.

It remains a very interesting outstanding problem to go beyond the truncation to two-impurity intermediate states on either side of the plane-wave gauge/string theory duality. On the gauge theory side, this would apparently require diagonalizing $\Delta - J$ at finite λ within the space of all gauge theory operators, and then taking the $\lambda, J \to \infty$ limits to decouple the non-BMN operators.

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A. Useful sums

For $n, p > 0$ and $r \in \{1, 2\}$ we obtain from (3.10), (3.11) and (3.12) the identities

$$\begin{aligned}
\sum_{q\geq 0} \overline{N}^{(3s)}_{nq}\overline{N}^{(3s)}_{pq} &= -\frac{(\delta_{s,1}(-1)^{n+p}+\delta_{s,2})}{\pi}\left[\frac{\sin\left(\pi(n-p)\frac{\alpha_{(s)}}{\alpha_{(3)}}\right)}{n-p}+\frac{\sin\left(\pi(n+p)\frac{\alpha_{(s)}}{\alpha_{(3)}}\right)}{n+p}\right], \\
\sum_{q>0} \overline{N}^{3s}_{-n-q}\overline{N}^{(3s)}_{-p-q} &= -\frac{(\delta_{s,1}(-1)^{n+p}+\delta_{s,2})}{\pi}\left[\frac{\sin\left(\pi(n-p)\frac{\alpha_{(s)}}{\alpha_{(3)}}\right)}{n-p}-\frac{\sin\left(\pi(n+p)\frac{\alpha_{(s)}}{\alpha_{(3)}}\right)}{n+p}\right].
\end{aligned} \tag{A.1}$$

B. The matrix B

The matrix B is given by [19]

$$B_{nm} = \begin{cases} 0 & n = 0 \text{ if } m = 0 \\ \frac{1}{3}+\frac{10}{u^2} & \text{if } n = m \neq 0 \\ -\frac{15}{2u^2} & \text{if } n = -m \neq 0 \\ \frac{6}{uv}+\frac{2}{(u-v)^2} & \text{all other cases,} \end{cases} \tag{B.1}$$

where $u = 2\pi m$, $v = 2\pi n$.

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